

Form factors of the XXZ Heisenberg spin- $\frac{1}{2}$ finite chain

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Abstract

Form factors for local spin operators of the XXZ Heisenberg spin- $\frac{1}{2}$ finite chain are computed. Representation theory of Drinfel'd twists for the quantum affine algebra $\mathcal{U}_q(\hat{sl}_2)$ in finite dimensional modules is used to calculate scalar products of Bethe states (leading to Gaudin formula) and to solve the quantum inverse problem for local spin operators in the finite chain. Hence, we obtain the representation of the n-spin correlation functions in terms of expectation values (in ferromagnetic reference state) of the operator entries of the quantum monodromy matrix satisfying Yang-Baxter algebra. This leads to the direct calculation of the form factors of the XXZ Heisenberg spin- $\frac{1}{2}$ finite chain as determinants of usual functions of the parameters of the model. A two-point correlation function for adjacent sites is also derived using similar techniques.

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1 Introduction

One of the most challenging problems in the theory of low dimensional quantum integrable models [1, 2, 3, 4, 5, 6], after finding the spectrum and eigenstates of the corresponding Hamiltonians, is to construct exact and manageable expressions of their form factors and correlation functions. This is a fundamental problem both to enlarge the range of applications of these models in the realm of condensed matter physics and to better understand their underlying mathematical structures. Until recently, only very few models were known for which correlation functions can be computed exactly. Typical examples are the Ising model (related to free fermions) [7, 8, 9] and conformal field theories [10, 11].

Beyond these models, in the framework of integrable systems solvable by means of Bethe Ansatz [12, 13, 14, 2, 1, 3, 5], related to a Quantum Group structure [15, 16, 17, 18] and associated to an R-matrix solving the Yang-Baxter equation, one can distinguish at present essentially two different but complementary approaches that have been designed to deal with this problem.

One of them relies on the study of form factors and correlation functions of quantum integrable models directly in the infinite volume limit. The roots of this approach are twofold:

On the one hand, it comes from the study of analytic properties and bootstrap equations for the factorized S-matrices and form factors of integrable quantum field theories in infinite volume [19, 4]. Typical models here are the two-dimensional Sine-Gordon relativistic quantum field theory and the Non-Linear Schrödinger model. There it was realized that the set of equations satisfied by the form factors are closely related to the q-deformed Knizhnik-Zamolodchikov equations arising from representation theory of quantum affine algebras, and their q-deformed vertex operators [20, 21, 22, 23, 24, 25, 26, 27].

On the other hand, it uses the Corner Transfer Matrix introduced by Baxter [28, 29, 1, 30] in the context of integrable models of statistical mechanics, and the relation of its spectra to characters of affine Lie algebras [31]. Typical examples here are the six-vertex model and the XXZ Heisenberg spin- $\frac{1}{2}$ infinite chain. In such models, using very plausible hypothesis about the representation of the Hamiltonian as a central element of the corresponding quantum affine algebra (here $\mathcal{U}_q(\hat{sl}_2)$) in the infinite volume limit, the space of states is constructed in terms of highest weight modules of $\mathcal{U}_q(\hat{sl}_2)$ [32, 33, 6], the combinatorial aspects of this construction being related to the theory of crystal bases [21, 34]. Form factors and correlation functions are then described in terms of q-deformed vertex operators, leading via bosonization [35], to integral formulas for them. As a result of its algebraic formulation, very parallel to the one in conformal field theory, there was a rapid development of this approach, leading to rather explicit expressions for correlators and their short distance behaviour [6].

It should be mentioned however, that the application of this method seems more difficult for time or temperature dependent correlators, or, for quantum spin chains, if an external magnetic field is present, namely in situations where the infinite symmetry algebra is not as clearly identified as in the pure case. In these directions, one should cite [36, 37], where in particular a finite volume analysis of form factors has been undertaken.

The other approach, described essentially in the book [5], is based on the detailed analysis of the structure of Bethe eigenstates and in particular of their scalar products properties. One of the starting points of this approach is the Algebraic Bethe Ansatz (or Quantum Inverse Scattering) method [14, 2, 1, 5] and the derivation in this framework by Korepin of the Gaudin formula for the norm of Bethe eigenstates [38]. Then, to overcome the enormous combinatorial complexity due

to the structure of Bethe eigenstates, the two key ingredients of this method are on the one hand the so called “dual fields approach” [39] and on the other hand the determinant expression for the partition function with “domain wall” boundary conditions [40]. Using these auxiliary quantum “dual fields”, determinant representations of correlation functions are obtained [41, 42, 43, 5], containing however vacuum expectation values of these auxiliary “dual fields”, which cannot be eliminated in the final result. Hence, explicit expressions for the correlators cannot be obtained directly from this approach. Instead, the strategy is to embed these determinant formulas in systems of integrable integro-difference equations from which only large distance asymptotics of the correlation functions can be extracted from the resolution of (matrix) Riemann-Hilbert problems.

Let us nevertheless note here, that in simpler models, in particular at so-called free-fermion points, such as the XX0 model or the Non-Linear Schrödinger model at infinite coupling constant, more explicit results can be obtained [44, 45].

Although mainly restricted to the determination of these large distance asymptotics of the correlation functions, the very general formulation of this method allows one to apply it to a large variety of integrable models, and to correlation functions depending on time, temperature and eventually, in the case of spin chains, on an external magnetic field.

A more algebraic understanding of the Bethe Ansatz approach to correlation functions is certainly needed to avoid the combinatorial difficulties encountered in this method, in particular if one would like to obtain explicit expressions for the correlators, namely, without auxiliary “dual fields”.

In this direction one should mention [46] where Gauss decomposition of operators was used within the Gaudin model to produce an explicit determinant formula for the norm of Bethe states, or also [26] where the Gaudin formula follows from semi-classical asymptotics of the q -deformed Knizhnik-Zamolodchikov equations.

The present state of the problem is such that, despite the great advances we just briefly described, algebraic derivation of form factors and correlation functions in an explicit and manageable setting, even for the most elementary models such as the XXZ Heisenberg spin- $\frac{1}{2}$ *finite* chain, still poses a formidable problem.

The main motivation of this article is precisely to understand from a more algebraic point of view the Bethe Ansatz approach to correlation functions for finite systems, and to try eventually to relate the two above approaches in taking the thermodynamic limit. For that purpose, we will mainly concentrate on one of the a priori most elementary models in this context, the XXZ Heisenberg spin- $\frac{1}{2}$ *finite* chain. We will show how to compute explicit determinant formulas, namely in terms of usual functions of the parameters of the model and without any auxiliary “dual fields”, for the form factors of local spin operators (i.e. their matrix elements between any two Bethe eigenstates) and for the adjacent sites two-point correlation function. In fact, we will also obtain this result for the completely inhomogeneous XXZ Heisenberg spin- $\frac{1}{2}$ *finite* chain.

Our approach to form factors and correlation functions for this model decomposes into three main steps:

- i.* we compute representations for scalar products of an arbitrary Bethe eigenstate with any other state in terms of determinants of elementary functions of the parameters of the model.

- ii. we solve the quantum inverse problem for the completely inhomogeneous XXZ Heisenberg spin- $\frac{1}{2}$ *finite* chain, namely, we reconstruct the local spin operators at any site i on the chain in terms of the elements of the quantum monodromy matrix of the chain.
- iii. we combine these two results to obtain determinant formulas for the form factors of the local spin operators, and for an adjacent sites two-point correlation function.

The key ingredient of our method is the article [47]. There a factorizing Drinfel'd twist F was constructed and studied. This twist is associated to the N -fold tensor product of spin- $\frac{1}{2}$ (evaluation) representation of the quantum affine algebra $\mathcal{U}_q(\hat{\mathfrak{sl}}_2)$ associated to the completely inhomogeneous XXZ Heisenberg spin- $\frac{1}{2}$ chain of length N . It has been shown in particular that the change of basis in quantum space of states generated by this twist F is particularly convenient to study the structure of Bethe eigenstates in the framework of Algebraic Bethe Ansatz. The main explanation of this is certainly the fact that the F -basis determines a completely symmetric presentation of the monodromy matrix operator for the (inhomogeneous) chain, such that the action of the symmetry group is trivial in this basis.

As a result, while creation and annihilation operators of Bethe states, namely the operator matrix elements of the quantum monodromy matrix, $B(\lambda)$ and $C(\lambda)$, are in the original basis represented as huge sums, containing up to 2^N terms, each of them being a product of N spin operators along the chain, their representations in this new F -basis simplify drastically. Indeed, in this basis, *they are given as sums of only N terms, each of them being simply a local spin operator at some site i of the chain, dressed by a pure tensor product of diagonal operators acting on the other sites* (see section 2 and [47]).

This means that the F -basis already solves the combinatorial problem of describing creation and annihilation operators of Bethe states. Moreover, we will show in this paper how it also solves the combinatorial problem of describing Bethe eigenstates generated by products of creation operators $B(\lambda_k)$ on a reference (ferromagnetic) state. This will enable us to compute in an explicit way scalar products of a Bethe eigenstate with any other state. The result is obtained for the completely inhomogeneous XXZ Heisenberg spin- $\frac{1}{2}$ *finite* chain, as determinants of functions of the parameters of the model, and solves the above point (i).

Point (ii) is also solved by a careful study of the particularly simple structure of the quantum monodromy matrix in the F -basis. The reconstruction of local spin operators in terms of the operator matrix elements of the quantum monodromy matrix is then obtained in a basis-independent way.

Point (iii) uses only the combination of this two results and of some resummation formulas we will explain in the main text.

This article is organized as follows: in section 2 we recall the definition of the inhomogeneous XXZ Heisenberg spin- $\frac{1}{2}$ *finite* chain, and the algebraic ingredients we will use in the following, such as the quantum R -matrix and the associated quantum monodromy matrix. Further, we describe briefly formulas for the factorizing twist F from [47] and its essential properties to be used in this article. In particular, we give there the expression of the quantum monodromy matrix in the F -basis. In section 3 we derive an explicit formula for the scalar product of an arbitrary Bethe state with any other state. Details of the proofs for this section are contained in the three appendices at the end of this article. Section 4 is devoted to the solution of the quantum inverse problem for the local spin operators. Finally, the main results of our work concerning form factors of the local spin operators are presented in section 5. Conclusions and perspectives are given in section 6.

2 The XXZ Heisenberg spin- $\frac{1}{2}$ inhomogeneous finite chain

In this paper, we shall calculate form factors for the Heisenberg XXZ and XXX spin- $\frac{1}{2}$ chains of length N . The XXZ Heisenberg model is given by the following Hamiltonian:

$$H_{\text{XXZ}} = J \sum_{m=1}^N \left\{ \sigma_m^x \sigma_{m+1}^x + \sigma_m^y \sigma_{m+1}^y + \Delta (\sigma_m^z \sigma_{m+1}^z - 1) \right\}, \quad (2.1)$$

the particular case $\Delta = 1$ corresponding to the XXX chain. We impose here periodic boundary conditions.

Our method is based on the Algebraic Bethe Ansatz [14, 2], the central object of which is the quantum R -matrix. For the XXX and XXZ models it is of the form

$$R(\lambda, \mu) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & b(\lambda, \mu) & c(\lambda, \mu) & 0 \\ 0 & c(\lambda, \mu) & b(\lambda, \mu) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (2.2)$$

where

$$b(\lambda, \mu) = \frac{\varphi(\lambda - \mu)}{\varphi(\lambda - \mu + \eta)}, \quad (2.3)$$

$$c(\lambda, \mu) = \frac{\varphi(\eta)}{\varphi(\lambda - \mu + \eta)}, \quad (2.4)$$

with the function φ defined as

$$\varphi(\lambda) = \lambda \quad \text{in the XXX case,} \quad (2.5)$$

$$\varphi(\lambda) = \sinh(\lambda) \quad \text{in the XXZ case.} \quad (2.6)$$

The R -matrix is a linear operator in the tensor product of two two-dimensional linear spaces $V_1 \otimes V_2$, where each V_i is isomorphic to \mathbf{C}^2 , and depends generically on two spectral parameters λ_1 and λ_2 associated to these two vector spaces. It is denoted by $R_{12}(\lambda_1, \lambda_2)$. Such an R -matrix satisfies the Yang-Baxter equation,

$$R_{12}(\lambda_1, \lambda_2) R_{13}(\lambda_1, \lambda_3) R_{23}(\lambda_2, \lambda_3) = R_{23}(\lambda_2, \lambda_3) R_{13}(\lambda_1, \lambda_3) R_{12}(\lambda_1, \lambda_2), \quad (2.7)$$

the unitary condition (provided $b(\lambda_1, \lambda_2) \neq \pm c(\lambda_1, \lambda_2)$),

$$R_{12}(\lambda_1, \lambda_2) R_{21}(\lambda_2, \lambda_1) = \mathbf{1}, \quad (2.8)$$

and the crossing symmetry relation,

$$(\gamma \otimes \mathbf{1}) R_{12}(\lambda_1^s, \lambda_2) (\gamma \otimes \mathbf{1}) = R_{21}^{t_1}(\lambda_2, \lambda_1) \rho(\lambda_1, \lambda_2), \quad (2.9)$$

with $\rho(\lambda_1, \lambda_2)$ being a scalar function, and γ a 2×2 matrix such that $\gamma^2 = \mathbf{1}$, $\gamma^t = \pm \gamma$, the upperscript t_j meaning the usual transposition of matrices in the corresponding space (j). For the rational case,

$$\lambda_1^s = \lambda_1 - \eta, \quad \gamma = \sigma^y, \quad \rho(\lambda_1, \lambda_2) = \frac{\lambda_1 - \lambda_2 - \eta}{\lambda_1 - \lambda_2},$$

and for the trigonometric case,

$$\lambda_1^s = \lambda_1 - \eta + i\pi, \quad \gamma = \sigma^x, \quad \rho(\lambda_1, \lambda_2) = \frac{\sinh(\lambda_1 - \lambda_2 - \eta)}{\sinh(\lambda_1 - \lambda_2)},$$

where σ^x and σ^y are the standard Pauli matrices.

Identifying one of the two linear spaces in the R -matrix with the two-dimensional Hilbert space \mathcal{H}_n of $SU(2)$ spin- $\frac{1}{2}$ corresponding to the site n of the chain, it is possible to construct the quantum L -operator of the model at site n as

$$L_n(\lambda, \xi_n) = R_{0n}(\lambda, \xi_n), \quad (2.10)$$

where ξ_n is an arbitrary (inhomogeneity) parameter dependent on the site n . The subscripts mean here that R_{0n} acts on the tensor product $\mathbf{C}^2 \otimes \mathcal{H}_n$. The quantum monodromy matrix of the total chain defined as the ordered product of L -operators is given by

$$T_0(\lambda) \equiv T_{0,1\dots N}(\lambda; \xi_1, \dots, \xi_N) = R_{0N}(\lambda, \xi_N) \dots R_{01}(\lambda, \xi_1). \quad (2.11)$$

It can be represented in the first space 0 as a 2×2 matrix,

$$T(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix}, \quad (2.12)$$

whose matrix elements $A(\lambda) \equiv A_{1\dots N}(\lambda; \xi_1, \dots, \xi_N)$, $B(\lambda) \equiv B_{1\dots N}(\lambda; \xi_1, \dots, \xi_N)$, $C(\lambda) \equiv C_{1\dots N}(\lambda; \xi_1, \dots, \xi_N)$, $D(\lambda) \equiv D_{1\dots N}(\lambda; \xi_1, \dots, \xi_N)$ are linear operators on the quantum space of states of the chain $\mathcal{H} = \bigotimes_{n=1}^N \mathcal{H}_n$. Their commutation relations are given by the following relation on $\mathbf{C}^2 \otimes \mathbf{C}^2$:

$$R_{12}(\lambda, \mu) T_1(\lambda) T_2(\mu) = T_2(\mu) T_1(\lambda) R_{12}(\lambda, \mu), \quad (2.13)$$

with the usual tensor notations $T_1(\lambda) = T(\lambda) \otimes \text{Id}$ and $T_2(\mu) = \text{Id} \otimes T(\mu)$.

The monodromy matrix satisfies moreover the following (crossing symmetry) relation leading to the definition of the quantum determinant:

$$T_{0,1\dots N}(\lambda; \xi_1, \dots, \xi_N) \gamma_0 T_{0,1\dots N}^{t_0}(\lambda^s; \xi_1, \dots, \xi_N) \gamma_0 = \rho(\lambda; \xi_1, \dots, \xi_N) \mathbf{1}, \quad (2.14)$$

where $\rho(\lambda; \xi_1, \dots, \xi_N) = \prod_{i=1}^N \rho(\lambda, \xi_i)$, and γ_0 is the matrix γ of eq. (2.9) acting in space 0.

One also defines the transfer matrix $\mathcal{T}(\lambda)$ as the trace $A(\lambda) + D(\lambda)$ of the total monodromy matrix. Thanks to the Yang-Baxter equation and the invertibility of the R -matrix, the transfer matrices commute with each other for different values of the spectral parameter λ . For the homogeneous case where all parameters ξ_i are equal, the Hamiltonian (2.1) can be obtained in terms of the transfer matrix by means of trace identities.

The Algebraic Bethe Ansatz, which deals with the problem of diagonalizing simultaneously $\mathcal{T}(\lambda)$ for all values of λ , supposes the existence of a reference state $|0\rangle$, called pseudo-vacuum, such that

$$\begin{aligned} A(\lambda)|0\rangle &= a(\lambda)|0\rangle, \\ D(\lambda)|0\rangle &= d(\lambda)|0\rangle, \\ C(\lambda)|0\rangle &= 0, \\ B(\lambda)|0\rangle &\neq 0. \end{aligned} \quad (2.15)$$

For the XXX or XXZ model, the pseudo-vacuum is the completely ferromagnetic state with all the spins up, and $a(\lambda) = 1$, $d(\lambda) = \prod_{i=1}^N b(\lambda, \xi_i)$. Common eigenstates of the transfer matrices for different values of the spectral parameter λ are obtained as successive actions of operators B on the pseudo-vacuum $\prod_{j=1}^n B(\lambda_j)|0\rangle$, for any set of n spectral parameters $\{\lambda_j, 1 \leq j \leq n\}$ solution of Bethe equations

$$r(\lambda_k) \prod_{\substack{j=1 \\ j \neq k}}^n \frac{b(\lambda_k, \lambda_j)}{b(\lambda_j, \lambda_k)} = 1, \quad 1 \leq k \leq n, \quad (2.16)$$

with

$$r(\lambda) = \frac{a(\lambda)}{d(\lambda)}. \quad (2.17)$$

The corresponding eigenvalue for the transfer matrix $\mathcal{T}(\mu)$ is then

$$\tau(\mu, \{\lambda_j\}) = a(\mu) \prod_{j=1}^n b^{-1}(\lambda_j, \mu) + d(\mu) \prod_{j=1}^n b^{-1}(\mu, \lambda_j). \quad (2.18)$$

Let us now turn to the description of the key object we will use to compute scalar products of Bethe states and to solve the Quantum Inverse Problem for local spins, leading finally to the form factors formulas: the factorizing F -matrix associated to the above R -matrix.

The concept of factorizing F -matrices was defined in [47], following the concept of twists introduced by Drinfel'd in the theory of Quantum Groups [15]. To be essentially self-contained we briefly recall here their main properties and refer to [47] for more details and proofs.

Due to the Yang-Baxter equation and to the unitarity of the R -matrix associated to the XXX and XXZ models, for any integer n one can associate to any element σ of the symmetric group S_n of order n , a unique R -matrix $R_{1\dots n}^\sigma(\xi_1, \dots, \xi_n)$ constructed as some ordered product (depending on σ) of the elementary R -matrices $R_{ij}(\xi_i, \xi_j)$ (see [47]), such that

$$R_{1\dots n}^\sigma(\xi_1, \dots, \xi_n) T_{0,1\dots n}(\lambda; \xi_1, \dots, \xi_n) = T_{0,\sigma(1)\dots\sigma(n)}(\lambda; \xi_{\sigma(1)}, \dots, \xi_{\sigma(n)}) R_{1\dots n}^\sigma(\xi_1, \dots, \xi_n). \quad (2.19)$$

A factorizing F -matrix associated to a given elementary R matrix is an invertible matrix $F_{1\dots n}(\xi_1, \dots, \xi_n)$ satisfying the following relation for any element σ of S_n :

$$F_{\sigma(1)\dots\sigma(n)}(\xi_{\sigma(1)}, \dots, \xi_{\sigma(n)}) R_{1\dots n}^\sigma(\xi_1, \dots, \xi_n) = F_{1\dots n}(\xi_1, \dots, \xi_n). \quad (2.20)$$

In other words, such an F -matrix factories the corresponding R -matrix. Taking into account the fact that the parameters ξ_i are in one to one correspondence with the vector spaces V_i , we can adopt simplified notations such that

$$\begin{aligned} F_{1\dots n}(\xi_1, \dots, \xi_n) &= F_{1\dots n}, \\ F_{\sigma(1)\dots\sigma(n)}(\xi_{\sigma(1)}, \dots, \xi_{\sigma(n)}) &= F_{\sigma(1)\dots\sigma(n)}, \\ F_{1,2\dots n}(\xi_1; \xi_2 \dots, \xi_n) &= F_{1,2\dots n}. \end{aligned}$$

An explicit formula for a triangular F -matrix corresponding to the XXZ model has been constructed in [47]. It reads for any integer n ,

$$F_{1\dots n} = F_{2\dots n} F_{1,2\dots n}, \quad (2.21)$$

$$= F_{n-1\ n} F_{n-2,n-1\ n} \dots F_{1,23\dots n}, \quad (2.22)$$

where the partial F -matrices $F_{i,i+1\dots n}(\xi_i; \xi_{i+1}, \dots, \xi_n)$ are given in terms of the R -matrices as

$$F_{i,i+1\dots n}(\xi_i; \xi_{i+1}, \dots, \xi_n) = e_i^{(11)} + e_i^{(22)} R_{i,i+1\dots n}(\xi_i; \xi_{i+1}, \dots, \xi_n). \quad (2.23)$$

Here we have defined the partial R -matrices acting in $V_i \otimes \dots \otimes V_n$ as

$$R_{i,i+1\dots n}(\xi_i; \xi_{i+1}, \dots, \xi_n) = R_{in}(\xi_i, \xi_n) \dots R_{i\ i+1}(\xi_i, \xi_{i+1}), \quad (2.24)$$

and $e_i^{(kl)}$ is the elementary matrix $e^{(kl)}$ acting in space i , with matrix elements $e_{ab}^{(kl)} = \delta_{ak}\delta_{bl}$. The partial F -matrix $F_{0,1\dots N}(\lambda; \xi_1, \dots, \xi_N)$ has a useful expression as a 2×2 matrix in the first space 0 in terms of elements of the quantum monodromy matrix:

$$F_{0,1\dots N}(\lambda; \xi_1, \dots, \xi_N) = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ C_{1\dots N}(\lambda; \xi_1, \dots, \xi_N) & D_{1\dots N}(\lambda; \xi_1, \dots, \xi_N) \end{pmatrix}_{[0]}. \quad (2.25)$$

Let us note here two important properties we will use in the following. The first one can be derived directly from the above relations between R , T and F , leading to

$$\begin{aligned} F_{1\dots N}(\xi_1, \dots, \xi_N) T_{0,1\dots N}(\lambda; \xi_1, \dots, \xi_N) F_{1\dots N}^{-1}(\xi_1, \dots, \xi_N) &= \\ = F_{\sigma(1)\dots\sigma(N)}(\xi_{\sigma(1)}, \dots, \xi_{\sigma(N)}) T_{0,\sigma(1)\dots\sigma(N)}(\lambda; \xi_{\sigma(1)}, \dots, \xi_{\sigma(N)}) F_{\sigma(1)\dots\sigma(N)}^{-1}(\xi_{\sigma(1)}, \dots, \xi_{\sigma(N)}). \end{aligned}$$

Hence, it means that in the F -basis, the monodromy matrix \tilde{T} defined as

$$\tilde{T}_{0,1\dots N}(\lambda; \xi_1, \dots, \xi_N) = F_{1\dots N}(\xi_1, \dots, \xi_N) T_{0,1\dots N}(\lambda; \xi_1, \dots, \xi_N) F_{1\dots N}^{-1}(\xi_1, \dots, \xi_N), \quad (2.26)$$

is totally symmetric under any simultaneous permutations of the lattice sites i and of the corresponding inhomogeneity parameters ξ_i .

The second property, proved in [47], is as follows: for the XXZ- $\frac{1}{2}$ model, the quantum monodromy operator is a 2×2 matrix with entries A , B , C , D which are obtained as sums of 2^{N-1} operators which themselves are products of N local operators on the quantum chain. As an example, the B operator is given as

$$B_{1\dots N}(\lambda) = \sum_{i=1}^N \sigma_i^- \Omega_i + \sum_{i \neq j \neq k} \sigma_i^- (\sigma_j^- \sigma_k^+) \Omega_{ijk} + \text{higher terms}, \quad (2.27)$$

where σ^+ , σ^- and sgz are the standard Pauli matrices and the matrices Ω_i , Ω_{ijk} , are diagonal operators acting respectively on all sites but i , on all sites but i, j, k , and the higher order terms involve more and more exchange spin terms like $\sigma_j^- \sigma_k^+$. It means that the B operator returns one spin somewhere on the chain, this operation being however dressed non-locally and with non-diagonal operators by multiple exchange terms of the type $\sigma_j^- \sigma_k^+$.

So, whereas these formulas in the original basis are quite involved and cannot be used in direct computations, their expressions in the F -basis simplify drastically. From [47] we have

$$\begin{aligned}\tilde{D}_{1\dots N}(\lambda; \xi_1, \dots, \xi_N) &\equiv F_{1\dots N}(\xi_1, \dots, \xi_N) D_{1\dots N}(\lambda; \xi_1, \dots, \xi_N) F_{1\dots N}^{-1}(\xi_1, \dots, \xi_N) \\ &= \bigotimes_{i=1}^N \left(\begin{array}{cc} b(\lambda, \xi_i) & 0 \\ 0 & 1 \end{array} \right)_{[i]}.\end{aligned}\quad (2.28)$$

The operator \tilde{B} representing the operator B in the F -basis is given by

$$\tilde{B}_{1\dots N}(\lambda) = \sum_{i=1}^N \sigma_i^- c(\lambda, \xi_i) \bigotimes_{j \neq i} \left(\begin{array}{cc} b(\lambda, \xi_j) & 0 \\ 0 & b^{-1}(\xi_j, \xi_i) \end{array} \right)_{[j]}.\quad (2.29)$$

Similarly we have for the operator \tilde{C} ,

$$\tilde{C}_{1\dots N}(\lambda) = \sum_{i=1}^N \sigma_i^+ c(\lambda, \xi_i) \bigotimes_{j \neq i} \left(\begin{array}{ccc} b(\lambda, \xi_j) & b^{-1}(\xi_i, \xi_j) & 0 \\ & 0 & 1 \end{array} \right)_{[j]},\quad (2.30)$$

and the operator \tilde{A} can be obtained from quantum determinant relations (2.14).

We wish first to stress that the operators \tilde{A} , \tilde{B} , \tilde{C} , \tilde{D} satisfy the same quadratic commutation relations as A , B , C , D . Second, each of the operators \tilde{B} and \tilde{C} is reduced to an elementary sum on the sites of the chain of the corresponding spin operator at each site *dressed diagonally*, which is to be compared to their expressions in the original basis where they are given as sums of 2^N terms involving much more complicated operators.

It really means that the factorizing F -matrices we have constructed solve the combinatorial problem induced by the non-trivial action of the permutation group S_N given by the R -matrix. In the F -basis the action of the permutation group on the operators \tilde{A} , \tilde{B} , \tilde{C} , \tilde{D} is trivial. Moreover the operator $\tilde{A} + \tilde{D}$ which contains the Hamiltonian of the model together with the series of conserved quantities, is now a quasi-bi-local operator.

Further, it can be shown that the pseudo-vacuum state is left invariant, namely, it is an eigenvector of the total F -matrix with eigenvalue 1. Hence, in particular, the Algebraic Bethe Ansatz can be carried out also in the F -basis. For the scalar products of the quantum states of the model we have

$$\begin{aligned}\langle 0 | C(\lambda_1) \dots C(\lambda_n) B(\lambda_{n+1}) \dots B(\lambda_{2n}) | 0 \rangle &= \\ &= \langle 0 | \tilde{C}(\lambda_1) \dots \tilde{C}(\lambda_n) \tilde{B}(\lambda_{n+1}) \dots \tilde{B}(\lambda_{2n}) | 0 \rangle.\end{aligned}\quad (2.31)$$

Hence, thanks to these very simple expressions, a direct computation of Bethe eigenstates and of their scalar products in this F -basis is made possible, while it was completely hopeless in the original basis. There, only commutation relations between the operators A , B , C , D can be used, leading (see [5]) to very intricate sums over partitions.

We now end this section with some useful formulas making the computation of the F -matrices simpler, namely the expressions of the partial F -matrices in the F -basis. This will help us to solve the quantum inverse problem for the local spin operators.

Factorizing F -matrices are given in (2.22) as an ordered product of partial F -matrices like $F_{1,2\dots n}$. These object are constructed in terms of the R -matrix $R_{1,2\dots n}$. However this quantity is

highly non trivial to compute explicitly, since it involves in fact sums of 2^{n-1} terms. In contrast, the partial F -matrices in the F -basis can be obtained explicitly, while they also lead to the construction of factorizing F -matrices $F_{12\dots n}$. We have (using again simplified notations)

$$\begin{aligned} F_{1\dots n} &= \tilde{F}_{1,2\dots n} F_{2\dots n} \\ &= \tilde{F}_{1,2\dots n} \tilde{F}_{2,3\dots n} \dots F_{n-1n}, \end{aligned} \quad (2.32)$$

and the partial F -matrix $\tilde{F}_{1,2\dots n}$ reads as a 2×2 matrix in the first space 1:

$$\tilde{F}_{1,2\dots n}(\xi_1; \xi_2, \dots, \xi_n) = F_{2\dots n}(\xi_2, \dots, \xi_n) F_{1,2\dots n}(\xi_1; \xi_2, \dots, \xi_n) F_{2\dots n}^{-1}(\xi_2, \dots, \xi_n) \quad (2.33)$$

$$= \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \tilde{C}_{2\dots n}(\xi_1; \xi_2, \dots, \xi_n) & \tilde{D}_{2\dots n}(\xi_1; \xi_2, \dots, \xi_n) \end{pmatrix}_{[1]}. \quad (2.34)$$

It is a very simple object to compute from the formulas of this section. Hence using the F -basis we have also obtained a more explicit and elementary formula for the F -matrix itself.

3 Scalar products of Bethe states and the Gaudin formula

In this section we calculate the following scalar products of states constructed by the action of the operators $B(\lambda)$ on the pseudo-vacuum,

$$S_n(\{\mu_j\}, \{\lambda_k\}) = \langle 0 | \prod_{j=1}^n C(\mu_j) \prod_{k=1}^n B(\lambda_k) | 0 \rangle, \quad (3.1)$$

when one of the sets of parameters, for example $\{\lambda_k\}$, is a solution of Bethe equations. Hence the state $\prod_{k=1}^n B(\lambda_k) | 0 \rangle$ is supposed to be an eigenvector of the transfer matrix,

$$(A(\mu) + D(\mu)) \prod_{k=1}^n B(\lambda_k) | 0 \rangle = \tau(\mu, \{\lambda_k\}) \prod_{k=1}^n B(\lambda_k) | 0 \rangle, \quad (3.2)$$

with the eigenvalue

$$\tau(\mu, \{\lambda_k\}) = a(\mu) \prod_{k=1}^n b^{-1}(\lambda_k, \mu) + d(\mu) \prod_{j=1}^n b^{-1}(\mu, \lambda_k). \quad (3.3)$$

We will prove the following theorem:

Theorem 3.1. *Let $\{\lambda_1, \dots, \lambda_n\}$ be a solution of Bethe equations*

$$\frac{d(\lambda_j)}{a(\lambda_j)} \prod_{k \neq j} \frac{b(\lambda_k, \lambda_j)}{b(\lambda_j, \lambda_k)} = 1, \quad 1 \leq k \leq n,$$

and $\{\mu_1, \dots, \mu_n\}$ be an arbitrary set of parameters. Then the scalar product (3.1) can be represented as a ratio of two determinants

$$S_n(\{\mu_j\}, \{\lambda_k\}) = S_n(\{\lambda_k\}, \{\mu_j\}) = \frac{\det T(\{\mu_j\}, \{\lambda_k\})}{\det V(\{\mu_j\}, \{\lambda_k\})}, \quad (3.4)$$

of the following $n \times n$ matrices T and V :

$$T_{ab} = \frac{\partial}{\partial \lambda_a} \tau(\mu_b, \{\lambda_k\}), \quad V_{ab} = \frac{1}{\varphi(\mu_b - \lambda_a)}, \quad 1 \leq a, b \leq n. \quad (3.5)$$

Proof — Let us first note that the computation of the derivatives in the matrix T and of the determinant of the matrix V gives a formula obtained in [48]. However, the proofs proposed in [48, 49] are quite complicated and use some recursion relations for the scalar product or the dual field representation. Here we give a direct proof of this formula for XXX and XXZ models.

The usual approach to the scalar product developed in [38, 41, 42] is based on the commutation relations between the matrix elements of the monodromy matrix (operators $A(\lambda)$, $B(\lambda)$, $C(\lambda)$ and $D(\lambda)$). It leads to the recursion relations for the scalar products. Instead of it we use the explicit representations for these operators in the F -basis. Indeed, as the vacuum vector is invariant under the action of the operator F , the scalar product (3.1) can be rewritten in terms of the operators in the F -basis,

$$S_n = \langle 0 | \prod_{j=1}^n \tilde{C}(\mu_j) \prod_{k=1}^n \tilde{B}(\lambda_k) | 0 \rangle. \quad (3.6)$$

To perform the computation, it is convenient first to change the normalization of the operators $B(\lambda)$ and $C(\lambda)$:

$$\mathbb{B}(\lambda) = \frac{B(\lambda)}{d(\lambda)}, \quad \mathbb{C}(\lambda) = \frac{C(\lambda)}{d(\lambda)}. \quad (3.7)$$

We thus want to calculate the “renormalized” scalar product in the F -basis

$$\mathbb{S}_n = \langle 0 | \tilde{\mathbb{C}}(\mu_n) \dots \tilde{\mathbb{C}}(\mu_1) \tilde{\mathbb{B}}(\lambda_1) \dots \tilde{\mathbb{B}}(\lambda_n) | 0 \rangle, \quad (3.8)$$

in which we suppose $\{\lambda_k\}$ to be a solution of Bethe equations.

The idea is to insert in the scalar product complete sets of states $|i_1, \dots, i_m\rangle$ beyond each operator $\mathbb{C}(\lambda)$, where we denote by $|i_1, \dots, i_m\rangle$ the state with m spins down in the sites i_1, \dots, i_m and with $N-m$ spins up in the other sites. We are thus led to consider the intermediate functions

$$\mathbb{G}^{(m)}(\{\lambda_k\}, \mu_1, \dots, \mu_m, i_{m+1}, \dots, i_n) = \langle i_{m+1}, \dots, i_n | \tilde{\mathbb{C}}(\mu_m) \dots \tilde{\mathbb{C}}(\mu_1) \tilde{\mathbb{B}}(\lambda_1) \dots \tilde{\mathbb{B}}(\lambda_n) | 0 \rangle, \quad (3.9)$$

the last one being the scalar product,

$$\mathbb{G}^{(n)}(\{\lambda_k\}, \mu_1, \dots, \mu_n) = \mathbb{S}_n.$$

There is actually a very simple recursion relation between these functions,

$$\begin{aligned} \mathbb{G}^{(m)}(\{\lambda_k\}, \mu_1, \dots, \mu_m, i_{m+1}, \dots, i_n) &= \sum_{j \neq i_{m+1}, \dots, i_n} \langle i_{m+1}, \dots, i_n | \tilde{\mathbb{C}}(\mu_m) | j, i_{m+1}, \dots, i_n \rangle \times \\ &\times \mathbb{G}^{(m-1)}(\{\lambda_k\}, \mu_1, \dots, \mu_{m-1}, j, i_{m+1}, \dots, i_n), \end{aligned} \quad (3.10)$$

where the matrix elements of the operator $\tilde{\mathbb{C}}(\mu_m)$ can be easily calculated in the F -basis:

$$\langle i_{m+1}, \dots, i_n | \tilde{\mathbb{C}}(\mu) | j, i_{m+1}, \dots, i_n \rangle = \frac{\varphi(\eta)}{\varphi(\mu - \xi_j)} \prod_{a \neq j} b^{-1}(\xi_j, \xi_a) \prod_{l=m+1}^n \left(b(\mu, \xi_{i_l}) b(\xi_j, \xi_{i_l}) \right). \quad (3.11)$$

The function $\mathbb{G}^{(0)}$, defined as

$$\mathbb{G}^{(0)}(\{\lambda_k\}, i_1, \dots, i_n) = \langle i_1, \dots, i_n | \prod_{k=1}^n \tilde{\mathbb{B}}(\lambda_k) | 0 \rangle,$$

is closely related to the partition function of the six-vertex model with domain wall boundary conditions, which was initially given in [40]. In Appendix A we compute directly this partition function using the F -basis representation for the operators $B(\lambda)$ and $C(\lambda)$. The function $\mathbb{G}^{(0)}$ is then calculated in Appendix B:

$$\mathbb{G}^{(0)}(\{\lambda_\alpha\}, i_1, \dots, i_n) = \frac{\prod_{\alpha=1}^n \prod_{k=1}^n \varphi(\lambda_\alpha - \xi_{i_k} + \eta)}{\prod_{j>k} \varphi(\xi_{i_k} - \xi_{i_j}) \prod_{\alpha<\beta} \varphi(\lambda_\beta - \lambda_\alpha)} \det \mathcal{N}(\{\lambda_\alpha\}, \{\xi_{i_j}\}), \quad (3.12)$$

where the $n \times n$ matrix $\mathcal{N}(\{\lambda_\alpha\}, \{\xi_{i_j}\})$ is defined by

$$\mathcal{N}_{\alpha j} = \frac{\varphi(\eta)}{\varphi(\lambda_\alpha - \xi_{i_j} + \eta) \varphi(\lambda_\alpha - \xi_{i_j})}, \quad 1 \leq \alpha, j \leq n. \quad (3.13)$$

One should now perform the summation in the relation (3.10) and compute successively the functions $\mathbb{G}^{(m)}$. It can be done using some identities for rational functions. Detailed calculations are given in Appendix C. One finally obtains, both in the XXX and XXZ cases,

$$\mathbb{S}_n(\{\mu_j\}, \{\lambda_\alpha\}) = \mathbb{G}^{(n)}(\{\lambda_\alpha\}, \mu_1, \dots, \mu_n) = \frac{\det H(\{\lambda_\alpha\}, \{\mu_j\})}{\prod_{j>k} \varphi(\mu_k - \mu_j) \prod_{\alpha<\beta} \varphi(\lambda_\beta - \lambda_\alpha)}, \quad (3.14)$$

where the matrix elements of the $n \times n$ matrix $H(\{\lambda_\alpha\}, \{\mu_j\})$ are

$$H_{ab} = \frac{\varphi(\eta)}{\varphi(\lambda_a - \mu_b)} \left(r(\mu_b) \prod_{m \neq a} \varphi(\lambda_m - \mu_b + \eta) - \prod_{m \neq a} \varphi(\lambda_m - \mu_b - \eta) \right). \quad (3.15)$$

This formula was originally obtained in [48]. It can be rewritten in a very simple form in terms of eigenvalues of the transfer matrix (3.3),

$$S_n(\{\mu_j\}, \{\lambda_\alpha\}) = \frac{\prod_{\alpha=1}^n \prod_{j=1}^n \varphi(\mu_j - \lambda_\alpha)}{\prod_{j>k} \varphi(\mu_k - \mu_j) \prod_{\alpha<\beta} \varphi(\lambda_\beta - \lambda_\alpha)} \det T(\{\mu_j\}, \{\lambda_\alpha\}), \quad (3.16)$$

where the matrix T is a Jacobian,

$$T_{ab} = \frac{\partial}{\partial \lambda_a} \tau(\mu_b, \{\lambda_\alpha\}).$$

Using a well known formula,

$$\det V = \frac{\prod_{a < b} \varphi(\lambda_a - \lambda_b) \prod_{j < k} \varphi(\mu_k - \mu_j)}{\prod_{k=1}^n \prod_{a=1}^n \varphi(\mu_k - \lambda_a)}, \quad (3.17)$$

with the matrix V defined by (3.5), one can express the coefficient in (3.16) as a determinant and obtain finally the representation (3.4).

One should also mention that one can suppose from the beginning $\langle 0 | \prod_{\alpha=1}^n C(\lambda_\alpha)$ to be a Bethe state (instead of $\prod_{\alpha=1}^n B(\lambda_\alpha) | 0 \rangle$), make almost the same calculations, and obtain the following result:

$$\mathbb{S}_n(\{\lambda_\alpha\}, \{\mu_j\}) = \mathbb{S}_n(\{\mu_j\}, \{\lambda_\alpha\}). \quad (3.18)$$

□

It can be easily seen that taking the limit $\mu_a \rightarrow \lambda_a$, $a = 1, \dots, n$, in the expression (3.14), one obtains a very nice proof of the Gaudin formula for the square of the norm of the Bethe wave function, initially proved by Korepin [38],

$$\begin{aligned} \mathbb{N}_n &\equiv \langle 0 | \prod_{j=1}^n \mathbb{C}(\lambda_j) \prod_{k=1}^n \mathbb{B}(\lambda_k) | 0 \rangle, \\ &= \varphi^n(\eta) \prod_{\alpha \neq \beta} \frac{\varphi(\lambda_\alpha - \lambda_\beta + \eta)}{\varphi(\lambda_\alpha - \lambda_\beta)} \det \Phi'(\{\lambda_\alpha\}), \end{aligned} \quad (3.19)$$

where Φ' is a $n \times n$ matrix the elements of which are given by

$$\Phi'_{ab} = -\frac{\partial}{\partial \lambda_b} \ln \left(r(\lambda_a) \prod_{\substack{k=1 \\ k \neq a}}^n \frac{b(\lambda_a, \lambda_k)}{b(\lambda_k, \lambda_a)} \right). \quad (3.20)$$

4 Quantum inverse scattering problem for local spins and correlation functions

In the previous section we have calculated the scalar products of Bethe states using the F -basis. Our purpose is now to compute form factors of local spin operators. One possibility is to write them in the F -basis, which allows us to perform their calculation in the same way as for the scalar products. There is however a more simple and instructive way to proceed, which consists in solving the quantum inverse scattering problem for the local spin operators, that is expressing them only in terms of the operator entries A, B, C, D of the quantum monodromy matrix of the model.

4.1 Local spin operators in the F -basis and quantum inverse problem

As we just said, the first possible way to compute the form factors is to express the local spin operators in the F -basis. Note that it can be directly done for spin operators at the first or the last site of the chain. For example, due to the recursion relation $F_{1\dots N} = \tilde{F}_{1,2\dots N} F_{2\dots N}$ and to

the very simple form (2.34) of $\tilde{F}_{1,2\dots N}$, a direct computation of a product of 2×2 matrices in the space 1 gives the value for σ_1^- in the F -basis,

$$F_{1\dots N} \sigma_1^- F_{1\dots N}^{-1} = \tilde{D}_{1\dots N}(\xi_1; \xi_1, \dots, \xi_N) \sigma_1^-. \quad (4.1)$$

Similar expressions hold for σ_N^+ , σ_1^z and σ_N^z in the F -basis.

Thus, taking into account that \tilde{D} is totally symmetric, for a given site i of the chain this result can be simply translated into the following formula:

$$F_{i\dots N 1\dots i-1} \sigma_i^- F_{i\dots N 1\dots i-1}^{-1} = \tilde{D}_{1\dots N}(\xi_i) \sigma_i^-, \quad (4.2)$$

where we used the short notation $\tilde{D}_{1\dots N}(\lambda) \equiv \tilde{D}_{1\dots N}(\lambda; \xi_1, \dots, \xi_N)$ for $\lambda = \xi_i$. Hence, to calculate the operator σ_i^- in the F -basis $F_{1\dots N} \sigma_i^- F_{1\dots N}^{-1}$, one should evaluate the product of two “permuted” F -matrices $F_{1\dots N} F_{i\dots N 1\dots i-1}^{-1}$. It can be considered as the expression in the F -basis of some *propagator* $F_{i\dots N 1\dots i-1}^{-1} F_{1\dots N}$, for which the following result holds:

Lemma 1. *Let U_1^i be the propagator $F_{i\dots N 1\dots i-1}^{-1} F_{1\dots N}$ from site 1 to site i of the chain. It can be written into the two following forms:*

$$U_1^i = R_{i-1,i\dots N 1\dots i-2} \dots R_{2,3\dots N 1} R_{1,2\dots N}, \quad (4.3)$$

$$= \prod_{\alpha=1}^{i-1} (A_{1\dots N}(\xi_\alpha) + D_{1\dots N}(\xi_\alpha)). \quad (4.4)$$

Proof — Equality (4.3) comes from the factorizing property for the F -matrix (2.20) for a cyclic permutation σ [47]:

$$F_{\alpha\dots N 1\dots \alpha-1} R_{\alpha-1,\alpha\dots N 1\dots \alpha-2} = F_{\alpha-1\dots N 1\dots \alpha-2}. \quad (4.5)$$

Equality (4.4) follows from the identity

$$A_{1\dots N}(\xi_\alpha) + D_{1\dots N}(\xi_\alpha) = R_{\alpha,\alpha+1\dots N 1\dots \alpha-1}(\xi_\alpha; \xi_{\alpha+1}, \dots, \xi_{\alpha-1}). \quad (4.6)$$

The proof of (4.6) is based on the remark that $R_{0\alpha}(\xi_\alpha)$ is nothing but the permutation matrix $P_{0\alpha}$ of the spaces 0 and α . Writing $A_{1\dots N}(\xi_\alpha) + D_{1\dots N}(\xi_\alpha)$ as a trace in the auxiliary space V_0 , and making $P_{0\alpha}$ act on every factor, we obtain, thanks to the cyclicity of the trace,

$$\begin{aligned} A_{1\dots N}(\xi_\alpha) + D_{1\dots N}(\xi_\alpha) &= \text{tr}_0 (R_{0N}(\xi_\alpha) \dots R_{0\alpha+1}(\xi_\alpha) P_{0\alpha} R_{0\alpha-1}(\xi_\alpha) \dots R_{01}(\xi_\alpha)), \\ &= R_{\alpha\alpha-1}(\xi_\alpha) \dots R_{\alpha 1}(\xi_\alpha) R_{\alpha N}(\xi_\alpha) \dots R_{\alpha\alpha+1}(\xi_\alpha), \\ &= R_{\alpha,\alpha+1\dots N 1\dots \alpha-1}(\xi_\alpha; \xi_{\alpha+1}, \dots, \xi_{\alpha-1}), \end{aligned}$$

which ends the proof of lemma 1. Note that in (4.3), (4.4), all factors commute with each others.

□

Remark 4.1. *The propagator U_1^1 through the whole chain being the identity, let us notice that*

$$\prod_{\alpha=1}^N (A_{1\dots N}(\xi_\alpha) + D_{1\dots N}(\xi_\alpha)) = R_{N,1\dots N} \dots R_{1,2\dots N} = 1. \quad (4.7)$$

Thus, the inverse $(U_1^i)^{-1}$ of the propagator on a part of the chain is nothing but the propagator $U_{i+1}^1 = \prod_{\alpha=i+1}^N (A + D)(\xi_\alpha)$ on the remaining part.

Remark 4.2. *The action of the propagator consists in shifting the beginning of the chain from site 1 to site i . For an operator entry $X_{1\dots N}$ of the monodromy matrix ($X = A, B, C$ or D), it means that*

$$U_1^i X_{1\dots N} = X_{i\dots N1\dots i-1} U_1^i, \quad (4.8)$$

which, in terms of the monodromy matrix, can be written

$$U_1^i T_{0,1\dots N} = T_{0,i\dots N1\dots i-1} U_1^i. \quad (4.9)$$

The lemma 1 allows us to obtain the value of any local spin operator in the F -basis. For example, σ_i^- for a given site i of the chain becomes

$$F_{1\dots N} \sigma_i^- F_{1\dots N}^{-1} = \prod_{\alpha=1}^{i-1} (\tilde{A} + \tilde{D})(\xi_\alpha) \cdot \tilde{D}(\xi_i) \sigma_i^- \cdot \prod_{\alpha=1}^{i-1} (\tilde{A} + \tilde{D})^{-1}(\xi_\alpha). \quad (4.10)$$

Using this expression — and similar ones for σ_i^+ , σ_i^z — it is possible to compute directly the corresponding form factor by the same method as for scalar products.

Moreover, as the expressions of the operators $\tilde{B}(\lambda)$ and $\tilde{C}(\lambda)$ in the F -basis are quasilocal in terms of the σ_i^- or σ_i^+ , it is a way to solve the quantum inverse problem for local spin operators. Indeed, a direct calculus in the F -basis gives the identity

$$\tilde{D}(\xi_i) \sigma_i^- (\tilde{A} + \tilde{D})(\xi_i) = \tilde{B}(\xi_i), \quad (4.11)$$

which, with (4.10), leads to a new reconstruction of σ_i^- . There exists a straightforward proof of this last formula, that we expose in the next paragraph.

4.2 Reconstruction of local spin operators

In this paragraph, we prove an important result concerning the reconstruction of any local spin operator in the inhomogeneous spin chain in terms of elements of the monodromy matrix.

Theorem 4.1. *Local spin operators at a given site i of the inhomogeneous XXX or XXZ Heisenberg chain are given by*

$$\sigma_i^- = \prod_{\alpha=1}^{i-1} (A + D)(\xi_\alpha) \cdot B(\xi_i) \cdot \prod_{\alpha=i+1}^N (A + D)(\xi_\alpha), \quad (4.12)$$

$$\sigma_i^+ = \prod_{\alpha=1}^{i-1} (A + D)(\xi_\alpha) \cdot C(\xi_i) \cdot \prod_{\alpha=i+1}^N (A + D)(\xi_\alpha), \quad (4.13)$$

$$\sigma_i^z = \prod_{\alpha=1}^{i-1} (A + D)(\xi_\alpha) \cdot (A - D)(\xi_i) \cdot \prod_{\alpha=i+1}^N (A + D)(\xi_\alpha). \quad (4.14)$$

The proof of this theorem is a straightforward consequence of the following lemma when x_0 is respectively equal to σ_0^- , σ_0^+ and σ_0^z :

Lemma 2. *Let x_i be an operator acting on the quantum spin space V_i . We note x_0 the corresponding 2×2 matrix acting on the auxiliary space V_0 . They are related by the identity*

$$\text{tr}_0(x_0 R_{0,1\dots N}(\xi_i)) = \prod_{\alpha=1}^{i-1} (A + D)^{-1}(\xi_\alpha) \cdot x_i \cdot \prod_{\alpha=1}^i (A + D)(\xi_\alpha), \quad (4.15)$$

where the trace in the left hand side is taken on the matrix acting in V_0 .

Proof — Arguments used to prove the case $i = 1$ are quite similar to those of lemma 1:

$$\begin{aligned}\mathrm{tr}_0(x_0 R_{0,1\dots N}(\xi_1)) &= \mathrm{tr}_0(x_0 R_{0N}(\xi_1) \dots R_{02}(\xi_1) P_{01}), \\ &= x_1 R_{1N}(\xi_1) \dots R_{12}(\xi_1), \\ &= x_1 R_{1,2\dots N}(\xi_1),\end{aligned}$$

and we conclude with lemma 1.

To prove the general case, let us notice that in the F -basis $\tilde{R}_{0,1\dots N}(\xi_i)$ is completely symmetric into the spaces $1, \dots, N$, which enables us to consider that the chain begins with site i :

$$\begin{aligned}\mathrm{tr}_0(x_0 R_{0,1\dots N}(\xi_i)) &= F_{1\dots N}^{-1} \mathrm{tr}_0(x_0 \tilde{R}_{0,1\dots N}(\xi_i) F_{1\dots N}), \\ &= F_{1\dots N}^{-1} \mathrm{tr}_0(x_0 \tilde{R}_{0,i\dots N1\dots i-1}(\xi_i) F_{1\dots N}), \\ &= F_{1\dots N}^{-1} F_{i\dots N1\dots i-1} \mathrm{tr}_0(x_0 R_{0,i\dots N1\dots i-1}(\xi_i)) F_{i\dots N1\dots i-1}^{-1} F_{1\dots N}, \\ &= F_{1\dots N}^{-1} F_{i\dots N1\dots i-1} x_i (A + D)(\xi_i) F_{i\dots N1\dots i-1}^{-1} F_{1\dots N}.\end{aligned}$$

The value of the propagator $F_{i\dots N1\dots i-1}^{-1} F_{1\dots N}$ is given by lemma 1, which concludes the proof of lemma 2. \square

4.3 General formula for correlation functions

These results make it possible to write a general formula for any k -point spin-spin correlation function between two Bethe states for the inhomogeneous $\mathrm{XXX}-\frac{1}{2}$ or $\mathrm{XXZ}-\frac{1}{2}$ Heisenberg chain. Indeed, for any integer k and any subset $\{i_j\}_{1 \leq j \leq k}$ of $\{1, \dots, N\}$, with the convention $i_1 < i_2 < \dots < i_k$, the correlation function for spins at sites i_1, \dots, i_k between two Bethe states $\langle 0 | C(\mu_1) \dots C(\mu_{n_1})$ and $B(\lambda_1) \dots B(\lambda_{n_2}) | 0 \rangle$ can be written into the following form:

$$\begin{aligned}\langle 0 | C(\mu_1) \dots C(\mu_{n_1}) \sigma_{i_1}^{\epsilon_1} \sigma_{i_2}^{\epsilon_2} \dots \sigma_{i_k}^{\epsilon_k} B(\lambda_1) \dots B(\lambda_{n_2}) | 0 \rangle &= \\ &= \prod_{\alpha=1}^{i_1-1} \prod_{j=1}^{n_1} b^{-1}(\mu_j, \xi_\alpha) \cdot \prod_{\alpha=i_k+1}^N \prod_{j=1}^{n_2} b^{-1}(\lambda_j, \xi_\alpha) \times \\ &\quad \times \langle 0 | C(\mu_1) \dots C(\mu_{n_1}) \cdot X_{i_1}^{\epsilon_1}(\xi_{i_1}) \cdot \prod_{\alpha=i_1+1}^{i_2-1} (A + D)(\xi_\alpha) \cdot X_{i_2}^{\epsilon_2}(\xi_{i_2}) \dots \\ &\quad \dots \prod_{\alpha=i_{k-1}+1}^{i_k-1} (A + D)(\xi_\alpha) \cdot X_{i_k}^{\epsilon_k}(\xi_{i_k}) \cdot B(\lambda_1) \dots B(\lambda_{n_2}) | 0 \rangle, \quad (4.16)\end{aligned}$$

where ϵ_j , $1 \leq j \leq k$, takes the values $+$, $-$, or z , X^{ϵ_j} being equal respectively to C , B and $A - D$.

Hence we have reduced the problem of computing any correlation function of the XXZ model to a simpler problem written only in terms of the operator entries of the quantum monodromy matrix of the chain.

In the next section, we shall compute explicitly the form factors ($k = 1$) and the two-point correlation functions at adjacent sites ($k = 2$ and $i_1 = i_2 - 1$).

5 Form factors

We derive here explicit expressions for the form factors of the local spin operators for the finite inhomogeneous XXX and XXZ chains. More precisely, we calculate the matrix elements of the operators σ_m^+ , σ_m^- and σ_m^z between two Bethe eigenstates. We also give an expression for the simplest correlation function of two spin operators at adjacent sites.

5.1 Operators σ_m^- and σ_m^+

We begin with the calculation of the following one-point functions,

$$F_n^-(m, \{\mu_j\}, \{\lambda_k\}) = \langle 0 | \prod_{j=1}^{n+1} C(\mu_j) \sigma_m^- \prod_{k=1}^n B(\lambda_k) | 0 \rangle, \quad (5.1)$$

and

$$F_n^+(m, \{\lambda_k\}, \{\mu_j\}) = \langle 0 | \prod_{k=1}^n C(\lambda_k) \sigma_m^+ \prod_{j=1}^{n+1} B(\mu_j) | 0 \rangle, \quad (5.2)$$

where $\{\lambda_k\}_n$ and $\{\mu_j\}_{n+1}$ are solutions of Bethe equations. Using the results of the previous sections we prove here that they admit the following representations:

Proposition 5.1. *For two Bethe states with spectral parameters $\{\lambda_k\}_n$ and $\{\mu_j\}_{n+1}$, the matrix element of the operator σ_m^- can be represented as a determinant,*

$$F_n^-(m, \{\mu_j\}, \{\lambda_k\}) = \frac{\phi_{m-1}(\{\mu_j\})}{\phi_{m-1}(\{\lambda_k\})} \frac{\prod_{j=1}^{n+1} \varphi(\mu_j - \xi_m + \eta)}{\prod_{k=1}^n \varphi(\lambda_k - \xi_m + \eta)} \times \\ \times \frac{1}{\prod_{n+1 \geq k > j \geq 1} \varphi(\mu_k - \mu_j) \prod_{1 \leq \beta < \alpha \leq n} \varphi(\lambda_\beta - \lambda_\alpha)} \det_{n+1} H^-(m, \{\mu_j\}, \{\lambda_k\}), \quad (5.3)$$

where the coefficients $\phi_m(\{\lambda_k\})$ are

$$\phi_m(\{\lambda_k\}) = \prod_{k=1}^n \prod_{j=1}^m b^{-1}(\lambda_k, \xi_j), \quad (5.4)$$

and the $(n+1) \times (n+1)$ matrix H^- is defined as

$$H_{ab}^-(m) = \frac{\varphi(\eta)}{\varphi(\mu_a - \lambda_b)} \left(a(\lambda_b) \prod_{\substack{j=1 \\ j \neq a}}^{n+1} \varphi(\mu_j - \lambda_b + \eta) - d(\lambda_b) \prod_{\substack{j=1 \\ j \neq a}}^{n+1} \varphi(\mu_j - \lambda_b - \eta) \right) \quad \text{for } b < n+1, \quad (5.5)$$

$$H_{an+1}^-(m) = \frac{\varphi(\eta)}{\varphi(\mu_a - \xi_m + \eta) \varphi(\mu_a - \xi_m)}. \quad (5.6)$$

The matrix element $F_n^+(m, \{\lambda_k\}, \{\mu_j\})$ of the operator σ_m^+ admits a similar representation,

$$F_n^+(m, \{\lambda_k\}, \{\mu_j\}) = \frac{\phi_m(\lambda_k) \phi_{m-1}(\lambda_k)}{\phi_{m-1}(\mu_j) \phi_m(\mu_j)} F_n^-(m, \{\mu_j\}, \{\lambda_k\}). \quad (5.7)$$

Note that in the homogeneous limit $\xi_j = 0$, $j = 1, \dots, N$, the coefficients $\phi_m(\{\lambda_k\})$ are expressed in terms of the total momentum P of the state parameterized by $\{\lambda_k\}$,

$$\phi_m(\{\lambda_k\}) = \exp\{-iPm\},$$

with

$$P = \frac{i}{N} \sum_{k=1}^n \ln(r(\lambda_k)).$$

Proof — The proof of these representations is rather straightforward. As it was shown in previous section (theorem 4.1) the local operator σ_m^- can be expressed in terms of the transfer matrix and the operator $B(\xi_m)$ as

$$\sigma_m^- = \prod_{j=1}^{m-1} (A + D)(\xi_j) \cdot B(\xi_m) \cdot \prod_{j=m+1}^N (A + D)(\xi_j).$$

Since the Bethe states are eigenstates of the transfer matrix,

$$(A(\xi_j) + D(\xi_j)) \prod_{k=1}^n B(\lambda_k) |0\rangle = \left(\prod_{a=1}^n b^{-1}(\lambda_a, \xi_j) \right) \prod_{k=1}^n B(\lambda_k) |0\rangle, \quad (5.8)$$

the product of the operators $A(\xi_j) + D(\xi_j)$ contributes to the function $F_n^-(m, \{\mu_j\}, \{\lambda_k\})$ as a global factor:

$$F_n^-(m, \{\mu_j\}, \{\lambda_k\}) = \phi_m^{-1}(\{\lambda_k\}) \phi_{m-1}(\{\mu_j\}) \langle 0 | \prod_{j=1}^{n+1} C(\mu_j) B(\xi_m) \prod_{k=1}^n B(\lambda_k) | 0 \rangle. \quad (5.9)$$

Here we used a simple property of the solutions of Bethe equations,

$$\prod_{k=1}^n \prod_{j=1}^N b^{-1}(\lambda_k, \xi_j) = 1.$$

The right hand side of (5.9) thus reduces to a scalar product,

$$F_n^-(m, \{\mu_j\}, \{\lambda_k\}) = \phi_m^{-1}(\{\lambda_k\}) \phi_{m-1}(\{\mu_j\}) S_{n+1}(\{\mu_j\}, \{\xi_m, \lambda_1, \dots, \lambda_n\}), \quad (5.10)$$

which, $\{\mu_j\}$ being a solution of Bethe equations, can be computed by means of theorem 3.1. Hence

$$F_n^-(m, \{\mu_j\}, \{\lambda_k\}) = \phi_m^{-1}(\{\lambda_k\}) \phi_{m-1}(\{\mu_j\}) \frac{\det T_{n+1}(\{\mu_j\}, \{\xi_m, \lambda_1, \dots, \lambda_n\})}{\det V_{n+1}(\{\mu_j\}, \{\xi_m, \lambda_1, \dots, \lambda_n\})}, \quad (5.11)$$

where T and V are $(n+1) \times (n+1)$ matrices defined similarly as in theorem 3.1. Writing them explicitly one obtains the representation (5.3).

The form factor $F_n^+(m, \{\lambda_k\}, \{\mu_j\})$ can be calculated analogously using the representation for the operator σ_m^+ given by theorem 4.1. \square

5.2 Operator σ_m^z

We calculate here the matrix elements of the operator σ_m^z between two Bethe states,

$$F_n^z(m, \{\mu_j\}, \{\lambda_k\}) = \langle 0 | \prod_{j=1}^n C(\mu_j) \sigma_m^z \prod_{k=1}^n B(\lambda_k) | 0 \rangle.$$

We prove the following representation for this one-point function:

Proposition 5.2. *For two Bethe states with sets of spectral parameters $\{\lambda_k\}_n$ and $\{\mu_j\}_n$, the matrix element of the operator σ_m^z can be represented as a determinant,*

$$F_n^z(m, \{\mu_j\}, \{\lambda_k\}) = \frac{\phi_{m-1}(\{\mu_j\})}{\phi_{m-1}(\{\lambda_k\})} \prod_{j=1}^n \frac{\varphi(\mu_j - \xi_m + \eta)}{\varphi(\lambda_j - \xi_m + \eta)} \times \\ \times \frac{1}{\prod_{j>k} \varphi(\mu_k - \mu_j) \prod_{\alpha<\beta} \varphi(\lambda_\beta - \lambda_\alpha)} \det_n \left(H(\{\mu_j\}, \{\lambda_k\}) - 2P(m, \{\mu_j\}, \{\lambda_k\}) \right), \quad (5.12)$$

where $H(\{\mu_j\}, \{\lambda_k\})$ is the same matrix as for the determinant representation (3.14) of the scalar product,

$$H_{ab} = \frac{\varphi(\eta)}{\varphi(\mu_a - \lambda_b)} \left(a(\lambda_b) \prod_{j \neq a} \varphi(\mu_j - \lambda_b + \eta) - d(\lambda_b) \prod_{j \neq a} \varphi(\mu_j - \lambda_b - \eta) \right) \quad 1 \leq a, b \leq n,$$

and $P(m, \{\mu_j\}, \{\lambda_k\})$ is a matrix of rank one,

$$P_{ab}(m) = \varphi(\eta) \frac{\prod_{k=1}^n \varphi(\lambda_k - \lambda_b + \eta)}{\varphi(\mu_a - \xi_m) \varphi(\mu_a - \xi_m + \eta)} \quad 1 \leq a, b \leq n. \quad (5.13)$$

Proof — It is a bit more complicated than for σ^- or σ^+ . We use here a representation for σ_m^z which follows directly from remark 4.1 and theorem 4.1,

$$\sigma_m^z = 2 \prod_{j=1}^{m-1} (A + D)(\xi_j) \cdot A(\xi_m) \cdot \prod_{j=m+1}^N (A + D)(\xi_j) - I, \quad (5.14)$$

which leads to

$$F_n^z(m, \{\mu_j\}, \{\lambda_k\}) = 2 \phi_m^{-1}(\{\lambda_k\}) \phi_{m-1}(\{\mu_j\}) P_1(\xi_m, \{\mu_j\}, \{\lambda_k\}) - S(\{\mu_j\}, \{\lambda_k\}),$$

with

$$P_1(\xi_m, \{\mu_j\}, \{\lambda_k\}) = \langle 0 | \prod_{j=1}^n C(\mu_j) A(\xi_m) \prod_{k=1}^n B(\lambda_k) | 0 \rangle. \quad (5.15)$$

Let us now compute the function P_1 . It can be done using a well-known formula for the action of the operator $A(\xi)$ on an arbitrary state:

$$A(\xi_m) \prod_{k=1}^n B(\lambda_k) | 0 \rangle = \prod_{k=1}^n b^{-1}(\lambda_k, \xi_m) \prod_{k=1}^n B(\lambda_k) | 0 \rangle - \\ - \sum_{a=1}^N \frac{\varphi(\eta)}{\varphi(\lambda_a - \xi_m)} \left(\prod_{\substack{k=1 \\ k \neq a}}^n \frac{\varphi(\lambda_k - \lambda_a + \eta)}{\varphi(\lambda_k - \lambda_a)} \right) B(\xi_m) \prod_{\substack{k=1 \\ k \neq a}}^n B(\lambda_k) | 0 \rangle. \quad (5.16)$$

Hence P_1 reduces to a sum of scalar products, therefore to a sum of determinants according to theorem 3.1. It can be rewritten as a single determinant by means of the following formula for the determinant of the sum of two matrices one of which being of rank one. Indeed, if \mathcal{A} is an arbitrary $n \times n$ matrix and \mathcal{B} a rank one $n \times n$ matrix, the determinant of the sum $\mathcal{A} + \mathcal{B}$ is:

$$\det(\mathcal{A} + \mathcal{B}) = \det \mathcal{A} + \sum_{j=1}^n \det \mathcal{A}^{(j)},$$

where

$$\begin{aligned} \mathcal{A}_{ab}^{(j)} &= \mathcal{A}_{ab} \quad \text{for } b \neq j, \\ \mathcal{A}_{aj}^{(j)} &= \mathcal{B}_{aj}. \end{aligned}$$

Using this formula and also the orthogonality of two different Bethe states, one obtains the determinant representation (5.12). \square

Let us remark that in the homogeneous limit $\xi_j = 0$, $j = 1, \dots, N$, and for two identical Bethe eigenstates, the evident mean value of σ_m^z can be easily derived from these representations. Indeed (5.12) and (3.19) yield

$$s^z(n) \equiv \frac{\langle 0 | \prod_{j=1}^n C(\lambda_j) \sigma_m^z \prod_{k=1}^n B(\lambda_k) | 0 \rangle}{\langle 0 | \prod_{j=1}^n C(\lambda_j) \prod_{k=1}^n B(\lambda_k) | 0 \rangle} = \frac{\det(\Phi'(\{\lambda_j\}) - 2P'(\{\lambda_j\}))}{\det \Phi'(\{\lambda_j\})}, \quad (5.17)$$

where $\Phi'(\{\lambda_j\})$ is the Gaudin matrix given by (3.20) and $P'(\{\lambda_j\})$ a matrix of rank one:

$$P'_{ab} = -\frac{1}{N} \frac{\partial}{\partial \lambda_a} \ln r(\lambda_a).$$

Due to the following property of the Gaudin matrix,

$$\sum_{b=1}^n \Phi'_{ab} = -\frac{\partial}{\partial \lambda_a} \ln r(\lambda_a),$$

it can be rewritten as

$$s^z(n) = \det_n(I - 2U), \quad U_{ab} = \frac{1}{N},$$

which leads to the evident value of $s^z(n)$ (U being a rank one matrix):

$$s^z(n) = 1 - \frac{2n}{N}.$$

5.3 The correlation function of spins at adjacent sites

The next quantity to compute is the simplest two-point function, the correlator of two spins at adjacent sites,

$$F_n^{-+}(m, m+1, \{\mu_j\}, \{\lambda_k\}) \equiv \langle 0 | \prod_{j=1}^n C(\mu_j) \sigma_m^- \sigma_{m+1}^+ \prod_{k=1}^n B(\lambda_k) | 0 \rangle. \quad (5.18)$$

As usually $\{\lambda_k\}$ and $\{\mu_j\}$ are supposed to be solutions of Bethe equations. From theorem 4.1 this function can be written only in terms of the operators A , B , C and D . As previously, Bethe states being eigenstates for the propagator and the propagator from the first to the last site being equal to identity, we obtain a simple expression for this correlation function,

$$F_n^{-+}(m, m+1, \{\mu_j\}, \{\lambda_k\}) = \phi_{m-1}(\{\mu_j\}) \phi_{m+1}^{-1}(\{\lambda_k\}) \langle 0 | \prod_{j=1}^n C(\mu_j) B(\xi_m) C(\xi_{m+1}) \prod_{k=1}^n B(\lambda_k) | 0 \rangle. \quad (5.19)$$

It can then be reduced to a sum of scalar products by means of the commutation relations between the matrix elements of the monodromy matrix, namely,

$$C(\xi_{m+1}) \prod_{k=1}^n B(\lambda_k) | 0 \rangle = \sum_{a=1}^n M_a \prod_{\substack{k=1 \\ k \neq a}}^n B(\lambda_k) | 0 \rangle + \sum_{a \neq b} M_{ab} B(\xi_{m+1}) \prod_{\substack{k=1 \\ k \neq a, b}}^n B(\lambda_k) | 0 \rangle, \quad (5.20)$$

with the coefficients M_a and M_{ab} given by

$$M_a = \frac{\varphi(\eta)}{\varphi(\lambda_a - \xi_{m+1})} d(\lambda_a) \prod_{k \neq a}^n b^{-1}(\lambda_k, \xi_{m+1}) b^{-1}(\lambda_a, \lambda_k),$$

$$M_{ab} = -\frac{\varphi^2(\eta)}{\varphi(\lambda_a - \xi_{m+1}) \varphi(\lambda_b - \xi_{m+1})} d(\lambda_a) b^{-1}(\lambda_a, \lambda_b) \prod_{k \neq a, b}^n b^{-1}(\lambda_k, \lambda_b) b^{-1}(\lambda_a, \lambda_k). \quad (5.21)$$

Hence

$$F_n^{-+}(m, m+1, \{\mu_j\}, \{\lambda_k\}) = \phi_{m-1}(\{\mu_j\}) \phi_{m+1}^{-1}(\{\lambda_k\}) \times$$

$$\times \left(\sum_{a=1}^n M_a S_n(\{\mu_j\}, \{\xi_m, \lambda_1, \dots, \hat{\lambda}_a, \dots, \lambda_n\}) + \sum_{a \neq b} M_{ab} S_n(\{\mu_j\}, \{\xi_m, \xi_{m+1}, \lambda_1, \dots, \hat{\lambda}_a, \hat{\lambda}_b, \dots, \lambda_n\}) \right), \quad (5.22)$$

where the hat means that the corresponding parameter is not present in the set. Since $\{\mu_j\}$ is a solution of Bethe equations, the scalar products $S_n(\{\mu_j\}, \{\lambda_k\})$ can be represented as determinants of size n according to theorem 3.1. Therefore, two-point functions at adjacent sites are obtained as a sum of determinants of size n .

6 Conclusion

In this article we have computed explicit determinant representations for form factors of the XXZ Heisenberg spin- $\frac{1}{2}$ inhomogeneous finite chain. These determinants are simply given in terms of usual functions of the parameters of the model. Moreover, an adjacent sites correlator has also been determined using similar techniques. The knowledge of the factorizing F -matrices from [47] was an essential ingredient to achieve this goal. It also sheds some new light on the algebraic structure underlying the Bethe Ansatz approach to correlation functions. In particular, multi-point correlators have been expressed in terms of expectation values (on the ferromagnetic

reference state) of operator entries of the quantum monodromy matrix: this was the result of explicitly solving the quantum inverse problem for local spin operators at any site of the chain.

Let us stress also that this method allowed us to give a very direct proof of the scalar product formula between a Bethe eigenstate and an arbitrary state generated by the successive actions of the operators B . This formula is beautiful but very mysterious, since it involves the Jacobian of the eigenvalues of the transfer matrix with respect to the parameters of the Bethe states. Although the proof is very transparent, we do not know a satisfactory a priori explanation of it, and one feels that there should be a more direct understanding of this formula.

What remains to be done, at least for the form factors we have computed, is to describe their thermodynamic limit and to compare the obtained results to those following the approach [6]. This will be done in a forthcoming publication, for the spontaneous magnetization of the XXZ chain.

Another interesting question concerns the higher spin Heisenberg models. To deal with these more general cases, one would like to have at disposal the analogue of the factorizing F -matrices, but here for the higher (fused) R -matrices. This question is now under study.

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Appendix A

We give in this appendix the determinant representation for the partition function of the six-vertex model with domain wall boundary conditions, initially obtained in [40], and propose a direct proof for it, using the F -basis.

The partition function of the six-vertex model with domain wall boundary conditions corresponds to a special case of scalar product for the XXX or XXZ 1/2 spin chain:

$$Z_N(\{\lambda_\alpha\}, \{\xi_j\}) = \left\{ \prod_{j=1}^N \downarrow_j \right\} \left\{ B_{1\dots N}(\lambda_1; \xi_1, \dots, \xi_N) B_{1\dots N}(\lambda_2; \xi_1, \dots, \xi_N) \dots \right. \\ \left. \dots B_{1\dots N}(\lambda_N; \xi_1, \dots, \xi_N) \right\} \left\{ \prod_{j=1}^N \uparrow_j \right\}. \quad (\text{A.1})$$

It should be mentioned that the same partition function can be represented similarly as a matrix element of the products of operators $C(\lambda)$:

$$Z_N(\{\lambda_\alpha\}, \{\xi_j\}) = \left\{ \prod_{j=1}^N \uparrow_j \right\} \left\{ C_{1\dots N}(\xi_1; \lambda_1, \dots, \lambda_N) C_{1\dots N}(\xi_2; \lambda_1, \dots, \lambda_N) \dots \right. \\ \left. \dots C_{1\dots N}(\xi_N; \lambda_1, \dots, \lambda_N) \right\} \left\{ \prod_{j=1}^N \downarrow_j \right\}. \quad (\text{A.2})$$

Recursion relations for this function were obtained in [38]. It was shown that they define completely the function. The solution to these relations was given in [40] as a determinant both

in the XXX and XXZ cases. For the XXX case one has the following result:

$$Z_N(\{\lambda_\alpha\}, \{\xi_j\}) = \frac{\prod_{j=1}^N \prod_{\alpha=1}^N (\lambda_\alpha - \xi_j)}{\prod_{j>k} (\xi_k - \xi_j) \prod_{\alpha>\beta} (\lambda_\alpha - \lambda_\beta)} \det \mathcal{N}(\{\lambda_\alpha\}, \{\xi_j\}) , \quad (\text{A.3})$$

where $\mathcal{N}(\{\lambda_\alpha\}, \{\xi_j\})$ is the $N \times N$ matrix given by

$$\mathcal{N}_{\alpha j} = \frac{\eta}{(\lambda_\alpha - \xi_j + \eta)(\lambda_\alpha - \xi_j)} . \quad (\text{A.4})$$

In the XXZ case, the expression is similar:

$$Z_N(\{\lambda_\alpha\}, \{\xi_j\}) = \frac{\prod_{j=1}^N \prod_{\alpha=1}^N \sinh(\lambda_\alpha - \xi_j)}{\prod_{j>k} \sinh(\xi_k - \xi_j) \prod_{\alpha>\beta} \sinh(\lambda_\alpha - \lambda_\beta)} \det \mathcal{N}^{\text{xxz}}(\{\lambda_\alpha\}, \{\xi_j\}) , \quad (\text{A.5})$$

where $\mathcal{N}^{\text{xxz}}(\{\lambda_\alpha\}, \{\xi_j\})$ is the $N \times N$ matrix given by

$$\mathcal{N}_{\alpha j}^{\text{xxz}} = \frac{\sinh \eta}{\sinh(\lambda_\alpha - \xi_j + \eta) \sinh(\lambda_\alpha - \xi_j)} . \quad (\text{A.6})$$

We give now an explicit derivation of these representations, based on direct calculations in the F -basis. Indeed, using the expression of operator \tilde{B} (or \tilde{C}), one obtains a new recursion formula for the partition function, which corresponds to a development of the determinant in (A.3) or (A.5). Explicit calculations are performed here in the XXX case, but they are quite similar in the XXZ case.

More precisely, as the state $\{\prod_{j=1}^N \uparrow_j\}$ (respectively $\{\prod_{j=1}^N \downarrow_j\}$) is invariant under the left-action of $F_{1\dots N}$ (respectively the right-action of $F_{1\dots N}^{-1}$), the formula (A.1) can be directly written in the F -basis:

$$Z_N = \left\{ \prod_{j=1}^N \downarrow_j \right\} \left\{ \tilde{B}_{1\dots N}(\lambda_1; \xi_1, \dots, \xi_N) \dots \tilde{B}_{1\dots N}(\lambda_N; \xi_1, \dots, \xi_N) \right\} \left\{ \prod_{j=1}^N \uparrow_j \right\} ,$$

and, using the expression (2.29), we make $\tilde{B}_{1\dots N}(\lambda_N; \xi_1, \dots, \xi_N)$ act on the state $\{\prod_{j=1}^N \uparrow_j\}$, in order to obtain a recursion relation for Z_N :

$$Z_N = \sum_{i=1}^N c(\lambda_N, \xi_i) \left(\prod_{\substack{j=1 \\ j \neq i}}^N b(\lambda_N, \xi_j) \right) \left\{ \prod_{j=1}^N \downarrow_j \right\} \left\{ \tilde{B}_{1\dots N}(\lambda_1) \dots \tilde{B}_{1\dots N}(\lambda_{N-1}) \right\} \left\{ \left(\prod_{\substack{j=1 \\ j \neq i}}^N \uparrow_j \right) (\downarrow_i) \right\} .$$

We thus extract, for each term i of the sum, the action (which is now diagonal for $(\sigma_i^-)^2 = 0$) on space i of the other operators $\tilde{B}_{1\dots N}(\lambda_\alpha)$, $1 \leq \alpha \leq N-1$, which leads to an extra numerical factor:

$$\begin{aligned} Z_N = & \sum_{i=1}^N c(\lambda_N, \xi_i) \left(\prod_{j \neq i} b(\lambda_N, \xi_j) b^{-1}(\xi_i, \xi_j) \right) \times \\ & \times \left\{ \prod_{j \neq i} \downarrow_j \right\} \left\{ \tilde{B}_{1\dots i-1 i+1 \dots N}(\lambda_1; \xi_1, \dots, \xi_{i-1}, \xi_{i+1}, \dots, \xi_N) \dots \right. \\ & \left. \dots \tilde{B}_{1\dots i-1 i+1 \dots N}(\lambda_{N-1}; \xi_1, \dots, \xi_{i-1}, \xi_{i+1}, \dots, \xi_N) \right\} \left\{ \prod_{j \neq i} \uparrow_j \right\} . \end{aligned}$$

Indeed, as the number of operators \tilde{B} was formerly equal to the number of sites on the chain, and as $(\sigma_j^-)^2 = 0$, the product $\prod_{j=1}^N \sigma_j^-$ appears in all the non-zero terms in the development of the product of \tilde{B} .

We have obtained the following recursion formula for Z_N :

$$Z_N(\{\lambda_\alpha\}_{1 \leq \alpha \leq N}, \{\xi_j\}_{1 \leq j \leq N}) = \sum_{i=1}^N c(\lambda_N, \xi_i) \left(\prod_{\substack{j=1 \\ j \neq i}}^N b(\lambda_N, \xi_j) b^{-1}(\xi_i, \xi_j) \right) Z_{N-1}(\{\lambda_\alpha\}_{\alpha \neq N}, \{\xi_j\}_{j \neq i}). \quad (\text{A.7})$$

This corresponds actually to the last line development of the determinant in the formula

$$Z_N(\{\lambda_\alpha\}, \{\xi_j\}) = \frac{\prod_{j=1}^N \prod_{\alpha=1}^N (\lambda_\alpha - \xi_j)}{\prod_{j>k} (\xi_k - \xi_j) \prod_{\alpha>\beta} (\lambda_\alpha - \lambda_\beta)} \det \hat{\mathcal{N}},$$

where $\hat{\mathcal{N}}$ is the matrix obtained from \mathcal{N} by adding to the last line L_N the linear combination of the other lines $\sum_{\beta=1}^{N-1} f_\beta L_\beta$, with coefficients

$$f_\beta = - \prod_{k=1}^N \frac{\lambda_\beta - \xi_k + \eta}{\lambda_N - \xi_k + \eta} \cdot \prod_{\substack{\alpha=1 \\ \alpha \neq \beta}}^{N-1} \frac{\lambda_N - \lambda_\alpha}{\lambda_\beta - \lambda_\alpha}.$$

In fact, this development leads to the following recursion formula for Z_N :

$$\begin{aligned} Z_N = \sum_{i=1}^N \frac{\eta}{\lambda_N - \xi_i + \eta} \left(\prod_{\substack{k=1 \\ k \neq i}}^N \frac{\lambda_N - \xi_k}{\lambda_N - \xi_k + \eta} \right) \left(\prod_{\substack{k=1 \\ k \neq i}}^N \frac{1}{\xi_k - \xi_i} \right) \left(\prod_{\beta=1}^{N-1} \frac{\lambda_\beta - \xi_i}{\lambda_N - \lambda_\beta} \right) \left\{ \prod_{\substack{k=1 \\ k \neq i}}^N (\lambda_N - \xi_k + \eta) \right. \\ \left. - \sum_{\beta=1}^{N-1} \prod_{\substack{k=1 \\ k \neq i}}^N (\lambda_\beta - \xi_k + \eta) \frac{\lambda_N - \xi_i}{\lambda_\beta - \xi_i} \left(\prod_{\substack{\alpha=1 \\ \alpha \neq \beta}}^{N-1} \frac{\lambda_N - \lambda_\alpha}{\lambda_\beta - \lambda_\alpha} \right) \right\} Z_{N-1}(\{\lambda_\alpha\}_{\alpha \neq N}, \{\xi_j\}_{j \neq i}), \end{aligned}$$

and one can easily see that

$$\begin{aligned} \prod_{\substack{j=1 \\ j \neq i}}^N (\xi_i - \xi_j + \eta) = \left(\prod_{\beta=1}^{N-1} \frac{\xi_i - \lambda_\beta}{\lambda_N - \lambda_\beta} \right) \left\{ \prod_{\substack{k=1 \\ k \neq i}}^N (\lambda_N - \xi_k + \eta) \right. \\ \left. - \sum_{\beta=1}^{N-1} \prod_{\substack{k=1 \\ k \neq i}}^N (\lambda_\beta - \xi_k + \eta) \frac{\lambda_N - \xi_i}{\lambda_\beta - \xi_i} \left(\prod_{\substack{\alpha=1 \\ \alpha \neq \beta}}^{N-1} \frac{\lambda_N - \lambda_\alpha}{\lambda_\beta - \lambda_\alpha} \right) \right\}, \end{aligned}$$

equality between two polynomials of degree $N - 1$ in ξ_i , which can be proved at the N points $\xi_i = \lambda_\alpha$, $1 \leq \alpha \leq N$.

Thus, as \mathcal{N} et $\hat{\mathcal{N}}$ have the same determinant, this finishes our proof of the formula (A.3) for the partition function.

Appendix B

For the calculation of scalar products and correlation functions in the F -basis, we need determinant representations for the following functions

$$G_B^{(0)}(\{\lambda_k\}, i_1, \dots, i_n) \equiv \langle i_1, \dots, i_n | \prod_{k=1}^n \tilde{B}(\lambda_k) | 0 \rangle, \quad (\text{B.1})$$

$$G_C^{(0)}(\{\mu_l\}, i_1, \dots, i_n) \equiv \langle 0 | \prod_{l=1}^n \tilde{C}(\mu_l) | i_1, \dots, i_n \rangle. \quad (\text{B.2})$$

Such representations can be easily obtained from the one of the partition function Z_n , by calculating explicitly the action of the operators $\tilde{B}_{1\dots N}(\lambda_\alpha)$ or $\tilde{C}_{1\dots N}(\mu_\alpha)$ in the sites which do not belong to $\{i_1, \dots, i_n\}$. Indeed, in the expression for the operator $B(\lambda)$ in the F -basis

$$\tilde{B}_{1\dots N}(\lambda_\alpha) = \sum_{i=1}^N \sigma_i^- c(\lambda_\alpha, \xi_i) \otimes_{k \neq i} \begin{pmatrix} b(\lambda_\alpha, \xi_k) & 0 \\ 0 & b^{-1}(\xi_k, \xi_i) \end{pmatrix}_{[k]}, \quad (\text{B.3})$$

only the terms corresponding to a $\sigma_{i_k}^-$, where k belongs to $\{1, \dots, n\}$, give a non-zero contribution to the function $G_B^{(0)}(\{\lambda_k\}, i_1, \dots, i_n)$. It means that the operators $\tilde{B}_{1\dots N}(\lambda_\alpha)$, $1 \leq \alpha \leq n$, act as diagonal ones on all the spaces except i_1, \dots, i_n . Thus, we can extract the action of $\prod_{\alpha=1}^n \tilde{B}_{1\dots N}(\lambda_\alpha)$ on all the sites but i_1, \dots, i_n as a global factor:

$$\begin{aligned} G_B^{(0)}(\{\lambda_k\}, i_1, \dots, i_n) &= \left(\prod_{\alpha=1}^n \prod_{\substack{k=1 \\ k \neq i_1, \dots, i_n}}^N b(\lambda_\alpha, \xi_k) \right) Z_n(\{\lambda_\alpha\}, \{\xi_{i_j}\}) \\ &= \left(\prod_{\alpha=1}^n \prod_{k=1}^N b(\lambda_\alpha, \xi_k) \right) \frac{\prod_{\alpha=1}^n \prod_{k=1}^n \varphi(\lambda_\alpha - \xi_{i_k} + \eta)}{\prod_{j>k} \varphi(\xi_{i_j} - \xi_{i_k}) \prod_{\alpha<\beta} \varphi(\lambda_\alpha - \lambda_\beta)} \det \mathcal{N}(\{\lambda_\alpha\}, \{\xi_{i_j}\}), \end{aligned} \quad (\text{B.4})$$

where $\mathcal{N}(\{\lambda_\alpha\}, \{\xi_{i_j}\})$ is the $n \times n$ matrix defined by (A.4) or (A.6).

Using the representation of the operator $C(\mu)$ in the F -basis one can analogously obtain a representation for the function $G_C^{(0)}(\{\mu_l\}, i_1, \dots, i_n)$ (B.2). Both in the XXX and XXZ cases it is given by

$$G_C^{(0)}(\{\mu_l\}, i_1, \dots, i_n) = \left(\prod_{j=1}^n \prod_{\substack{k=1 \\ k \neq i_j}}^N b^{-1}(\xi_{i_j}, \xi_k) \right) \left(\prod_{\substack{j,k=1 \\ j \neq k}}^n b(\xi_{i_j}, \xi_{i_k}) \right) G_B^{(0)}(i_1, \dots, i_n, \{\mu_l\}). \quad (\text{B.5})$$

Appendix C

In this Appendix we calculate recursively the intermediate functions $\mathbb{G}^{(m)}$ (3.9). The detailed proof is written here only for the XXX case, but is quite similar in the XXZ case.

To compute the function $\mathbb{G}^{(1)}$ starting from the expression (3.12) for $\mathbb{G}^{(0)}$, one has to perform the summation in (3.10). Taking into account that only the first column of the matrix \mathcal{N} depends

on the parameter ξ_{i_1} , one can express $\mathbb{G}^{(1)}$ as follows

$$\mathbb{G}^{(1)}(\{\lambda_\alpha\}, \mu_1, i_2, \dots, i_n) = \frac{\prod_{\alpha=1}^n \prod_{k=2}^n (\lambda_\alpha - \xi_{i_k} + \eta)}{\prod_{n \geq j > k \geq 2} (\xi_{i_k} - \xi_{i_j}) \prod_{1 \leq \alpha < \beta \leq n} (\lambda_\beta - \lambda_\alpha)} \det \mathcal{N}^{(1)}(\{\lambda_\alpha\}, \mu_1, i_2, \dots, i_n), \quad (\text{C.1})$$

where the matrix $\mathcal{N}^{(1)}(\{\lambda_\alpha\}, \mu_1, i_2, \dots, i_n)$ is defined as

$$\mathcal{N}_{ab}^{(1)} = \frac{1}{\mu_1 - \xi_{i_b}} \mathcal{N}_{ab} \quad \text{for } b \geq 2, \quad (\text{C.2})$$

$$\mathcal{N}_{a1}^{(1)} = \prod_{k=2}^n (\mu_1 - \xi_{i_k} + \eta) \sum_{i_1=1}^n \frac{\eta}{\mu_1 - \xi_{i_1}} \frac{\eta}{\lambda_a - \xi_{i_1}} \frac{\prod_{m \neq a} (\lambda_m - \xi_{i_1} + \eta)}{\prod_{k=2}^n (\xi_{i_1} - \xi_{i_k} + \eta)} \prod_{j \neq i_1} b^{-1}(\xi_{i_1}, \xi_j). \quad (\text{C.3})$$

It should be mentioned that the summation in (C.3) can be taken over all the possible values of i_1 as the contributions of the terms $i_1 = i_j$, $j > 1$ are equal to zero. It is possible to calculate explicitly the sum in (C.3) using its analytical properties and the fact that $\{\lambda_\alpha\}$ is a solution of Bethe equations:

$$\begin{aligned} \sum_{i_1=1}^n \frac{\eta}{\mu_1 - \xi_{i_1}} \frac{\eta}{\lambda_a - \xi_{i_1}} \frac{\prod_{m \neq a} (\lambda_m - \xi_{i_1} + \eta)}{\prod_{k=2}^n (\xi_{i_1} - \xi_{i_k} + \eta)} \prod_{j \neq i_1} b^{-1}(\xi_{i_1}, \xi_j) &= \frac{H_{a1}}{\prod_{k=2}^n (\mu_1 - \xi_{i_k} + \eta)} \\ &+ \sum_{b=2}^n \frac{1}{\mu_1 - \xi_{i_b} + \eta} \cdot \frac{\prod_{m=1}^n (\lambda_m - \xi_{i_b})}{\prod_{\substack{j=1 \\ j \neq b}}^n (\xi_{i_b} - \xi_{i_j})} \cdot \frac{\eta}{(\lambda_a - \xi_{i_b})(\lambda_a - \xi_{i_b} + \eta)}, \end{aligned} \quad (\text{C.4})$$

where the function H_{ab} has the form

$$H_{ab} = \frac{\eta}{\lambda_a - \mu_b} \left(r(\mu_b) \prod_{m \neq a} (\lambda_m - \mu_b + \eta) - \prod_{m \neq a} (\lambda_m - \mu_b - \eta) \right), \quad (\text{C.5})$$

with

$$r(\mu) = \frac{a(\mu)}{d(\mu)} = \prod_{j=1}^N \frac{\mu - \xi_j + \eta}{\mu - \xi_j}.$$

Indeed, the left hand side of (C.4) is a rational function of μ_1 with simple poles at the points $\mu_1 = \xi_j$, $j = 1, \dots, N$, and its limit is zero when $\mu_1 \rightarrow \infty$. The right hand side is also a rational function of μ_1 , which has only simple poles and becomes zero when $\mu_1 \rightarrow \infty$. The residues of the r.h.s. at the points $\mu_1 = \xi_j$ are the same as in the l.h.s. Thus one should only prove that the r.h.s. has no other poles, namely when $\mu_1 = \lambda_a$ and $\mu_1 = \xi_{i_k} - \eta$, $k = 2, \dots, n$. One can easily see that the residues of the r.h.s. at the points $\mu_1 = \xi_{i_k} - \eta$ are equal to zero. As $\{\lambda_\alpha\}$ is a solution of Bethe equations, the residue at the point $\mu_1 = \lambda_a$ is also equal to zero. Therefore the l.h.s and r.h.s of (C.4) are rational functions having the same behavior when $\mu_1 \rightarrow \infty$, the same simple poles and the same residues in these poles, thus they are equal.

So, the matrix elements of the first column of the matrix $\mathcal{N}^{(1)}$ have the form

$$\mathcal{N}_{a1}^{(1)} = H_{a1} + \sum_{b=2}^n \alpha_b \mathcal{N}_{ab}^{(1)},$$

where α_b are coefficients which do not depend on a . Only the first term in this sum gives a nonzero contribution to the determinant of $\mathcal{N}^{(1)}$, which leads to the following representation for $\mathbb{G}^{(1)}$:

$$\mathbb{G}^{(1)}(\{\lambda_\alpha\}, \mu_1, i_2, \dots, i_n) = \frac{\prod_{\alpha=1}^n \prod_{k=2}^n (\lambda_\alpha - \xi_{i_k} + \eta)}{\prod_{n \geq j > k \geq 2} (\xi_{i_k} - \xi_{i_j}) \prod_{1 \leq \alpha < \beta \leq n} (\lambda_\beta - \lambda_\alpha)} \det \mathcal{G}^{(1)}(\{\lambda_\alpha\}, \mu_1, i_2, \dots, i_n), \quad (\text{C.6})$$

with the matrix $\mathcal{G}^{(1)}$ defined as

$$\begin{aligned} \mathcal{G}_{ab}^{(1)} &= \frac{1}{\mu_1 - \xi_{i_b}} \mathcal{N}_{ab} \quad \text{for } b \geq 2, \\ \mathcal{G}_{a1}^{(1)} &= H_{a1}. \end{aligned} \quad (\text{C.7})$$

Repeating this procedure, one obtains a general expression for all the functions $\mathbb{G}^{(m)}$:

$$\begin{aligned} \mathbb{G}^{(m)}(\{\lambda_k\}, \mu_1, \dots, \mu_m, i_{m+1}, \dots, i_n) &= \frac{\prod_{\alpha=1}^n \prod_{k=m+1}^n (\lambda_\alpha - \xi_{i_k} + \eta)}{\prod_{n \geq j > k \geq m+1} (\xi_{i_k} - \xi_{i_j}) \prod_{1 \leq \alpha < \beta \leq n} (\lambda_\beta - \lambda_\alpha)} \times \\ &\times \frac{1}{\prod_{m \geq j > k \geq 1} (\mu_k - \mu_j)} \det \mathcal{G}^{(m)}(\{\lambda_\alpha\}, \mu_1, \dots, \mu_m, i_{m+1}, \dots, i_n), \quad (\text{C.8}) \end{aligned}$$

with the matrix $\mathcal{G}^{(m)}(\{\lambda_\alpha\}, \mu_1, \dots, \mu_m, i_{m+1}, \dots, i_n)$ given by

$$\mathcal{G}_{ab}^{(m)} = \mathcal{N}_{ab} \prod_{j=1}^m \left(\frac{1}{\mu_j - \xi_{i_b}} \right) \quad \text{for } b > m, \quad (\text{C.9})$$

$$\mathcal{G}_{ab}^{(m)} = H_{ab} \quad \text{for } b \leq m. \quad (\text{C.10})$$

We prove (C.8)-(C.10) by induction. For $m = 1$ it coincides with (C.6). Let this representation be valid for $\mathbb{G}^{(m-1)}$. Combining it with (3.10) and (3.11) we obtain the following expression for $\mathbb{G}^{(m)}$:

$$\begin{aligned} \mathbb{G}^{(m)}(\{\lambda_k\}, \mu_1, \dots, \mu_m, i_{m+1}, \dots, i_n) &= \frac{\prod_{\alpha=1}^n \prod_{k=m+1}^n (\lambda_\alpha - \xi_{i_k} + \eta)}{\prod_{n \geq j > k \geq m+1} (\xi_{i_k} - \xi_{i_j}) \prod_{1 \leq \alpha < \beta \leq n} (\lambda_\beta - \lambda_\alpha)} \times \\ &\times \frac{1}{\prod_{m-1 \geq j > k \geq 1} (\mu_k - \mu_j)} \det \mathcal{N}^{(m)}(\{\lambda_\alpha\}, \mu_1, \dots, \mu_m, i_{m+1}, \dots, i_n), \quad (\text{C.11}) \end{aligned}$$

where the matrix $\mathcal{N}^{(m)}(\{\lambda_\alpha\}, \mu_1, \dots, \mu_m, i_{m+1}, \dots, i_n)$ is defined as

$$\begin{aligned}
\mathcal{N}_{ab}^{(m)} &= \mathcal{N}_{ab} \prod_{j=1}^m \left(\frac{1}{\mu_j - \xi_{i_b}} \right) && \text{for } b > m, \\
\mathcal{N}_{ab}^{(m)} &= H_{ab} && \text{for } b < m, \\
\mathcal{N}_{am}^{(m)} &= \prod_{k=m+1}^n (\mu_1 - \xi_{i_k} + \eta) \sum_{i_m=1}^n \frac{\eta}{\mu_m - \xi_{i_m}} \frac{\eta}{\lambda_a - \xi_{i_m}} \times \\
&\quad \times \frac{\prod_{l \neq a} (\lambda_l - \xi_{i_m} + \eta)}{\prod_{k=m+1}^n (\xi_{i_m} - \xi_{i_k} + \eta) \prod_{j=1}^{m-1} (\mu_j - \xi_{i_m})} \prod_{j \neq i_m} b^{-1}(\xi_{i_m}, \xi_j). \tag{C.12}
\end{aligned}$$

The sum in (C.12) can be computed the same way as in (C.4). One can prove using similar arguments that

$$\mathcal{N}_{am}^{(m)} = \frac{H_{am}}{\prod_{j=1}^{m-1} (\mu_j - \mu_m) \prod_{k=m+1}^n (\mu_m - \xi_{i_k} + \eta)} + \sum_{b=1}^{m-1} \beta_b^{(m)} H_{ab} + \sum_{b=m+1}^n \alpha_b^{(m)} \mathcal{N}_{ab}, \tag{C.13}$$

where $\alpha_b^{(m)}$ and $\beta_b^{(m)}$ are coefficients which do not depend on a . As only the first term gives a nonzero contribution to the determinant we obtain the representation (C.8).

Finally the scalar product is given by

$$\mathbb{S}_n(\{\mu_j\}, \{\lambda_\alpha\}) = \mathbb{G}^{(n)}(\{\lambda_\alpha\}, \mu_1, \dots, \mu_n) = \frac{\det H(\{\lambda_\alpha\}, \{\mu_j\})}{\prod_{j>k} (\mu_k - \mu_j) \prod_{\alpha<\beta} (\lambda_\beta - \lambda_\alpha)}, \tag{C.14}$$

with matrix elements of $H(\{\lambda_\alpha\}, \{\mu_j\})$ defined by (C.5).

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